Problems and Results in the Category of Topological Transformation Groups

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In this paper I shall give a brief survey of some work that has been done on equivariant general topology, as far as it is related with my own research. Thus, equivariant algebraic topology will not be considered, nor actions of Lie groups on manifolds and their classification. I will restrict myself to sketching some general problems; references to (partial) solutions will be given, but only very few of them will be discussed explicitly.

1. THE CATEGORY OF G-SPACES

Let G be an arbitrary topological group, fixed during the discussion. *A topological transformation group* (ttg) with *acting group* G, or more shortly, a *G-space,* is a pair $\langle X, \pi\rangle$ where X is a topological space and π: $(t, x) \mapsto \pi^t x$: $G \times X \rightarrow X$ is a continuous mapping such that $\pi^e = id_X$ (e is the unit element of G) and $\pi^{st} = \pi^s \circ \pi^t$ for all s, $t \in G$. In particular, $t \mapsto \pi^t$ is a (not necessarily injective) homeomorphism of the group G into the homeomorphism group of X.

If $\langle X, \pi\rangle$ and $\langle Y, \sigma\rangle$ are G-spaces, then a mapping $f: X \rightarrow Y$ is called *equivariant* whenever $f \circ \pi^t = \sigma^t \circ f$ for every $t \in G$. A continuous equivariant mapping will also be called a *morphism of G-spaces.* Consider the collection of all topological spaces and continuous mappings as a category, denoted by **Top**. If **K** is a subcategory of **Top**, then $\mathbf{K}^G$ will denote the category having as objects all G-spaces $\langle X, \pi\rangle$ with X an object in **K** and as morphisms all morphisms in **K** which are equivariant. The study of $\mathbf{Top}^G$ (and more general categories of G-spaces) was initiated in DE VRIES [31]. Basic is the observation that $\mathbf{Top}^G$ can be seen as the category of algebras over a suitable comonad in **Top**. If G is locally compact, then $\mathbf{Top}^G$ can also be seen as the category of co-algebras over a suitable monad in **Top**. This allows for a quite complete description of the category $\mathbf{Top}^G$ in terms of the category **Top**, the latter category being considered as known. For a survey of this theory, see also DE VRIES [32].

Since then a number of topological problems in the category $\mathbf{Top}^G$ has been studied by various authors, see DE VRIES [35,36] or ANTONYAN & SMIRNOV [9] and SMIRNOV [26] for surveys and further references. In these papers, general topology is regarded as a theory about $\mathbf{Top}^G$ with G the trivial group $\{e\}$, and results in this theory are generalised to the case of more general groups.

Sometimes, results are easily obtained by using the various functors between $\mathbf{Top}^G$ and $\mathbf{Top}$ studied in DE VRIES [31] and their preservation properties. For example, the results in KADIROV [14] about projective objects for certain classes of epimorphisms in $\mathbf{Top}^G$ and about projective hulls of G-spaces are not very surprising from this point of view. Interesting problems arise when those functors do *not* preserve or reflect the properties or constructions one is interested in. Generally speaking, there are the following three obstructions to a straightforward translation of results from $\mathbf{Top}$ to $\mathbf{Top}^G$:

(a) All mappings employed or to be constructed have to be equivariant. For example, if $\langle X, \pi\rangle$ and $\langle Y, \sigma\rangle$ are G-spaces, X metrizable and Y a compact convex subset of a locally convex topological vector space, A a closed invariant subset of X, and $f: A \to Y$ is a continuous equivariant mapping then f has a continuous extension $\tilde{f}$: $X \to Y$. But is there also an *equivariant* continuous extension? This would yield an equivariant version of Dugundji's extension theorem (or: a Dugundji's theorem for G-spaces). See Section 2 below.

(b) 'Categorically' constructed objects do not always belong to the desired subcategories of $\mathbf{Top}^G$. For example (related to the problem mentioned in (a) above), if K is an absolute extensor (injective object) in $\mathbf{Top}$ for a certain class of monomorphisms, then $\langle C_c(G, K), \rho\rangle$ (with $\rho^t f(s) = f(st)$ for $s, t \in G$ and $f \in C_c(G, K)$) is an equivariant extensor (injective object) for the corresponding class of monomorphisms in $\mathbf{Top}^G$ — this follows from categorical considerations — but $C_c(G, K)$ is never compact if G is not finite!

(c) All spaces employed or constructed have to be G-spaces. As to the question of which G-space should replace the interval $[0;1]$ (which is a very important object in $\mathbf{Top}$) and which mappings should play the role of continuous real valued functions see DE VRIES [39]. Another problem of this type: every Tychonov space X can be embedded in a compact Hausdorff space (e.g. in the Stone-Čeck compactification βX of X); if $\langle X, \pi\rangle$ is a G-space with X Tychonov, does there exist a G-space $\langle \tilde{X}, \tilde{\pi}\rangle$ with $\tilde{X}$ compact Hausdorff space in which X can be equivariantly embedded? (In general, π cannot be extended to a continuous (!) action of G on βX; see DE VRIES [36].) Similar questions can be asked with respect to completions. For compactifications, see Section 4, below; other problems of this type are considered in Section 3.

2. EQUIVARIANT EXTENSION OF MAPPINGS

Let me first mention the following result which is a version of a (by now) classical theorem from GLEASON [12] (see also PALAIS [24]):

THEOREM. *Let G be a compact topological group and let $\langle K, \alpha\rangle$ be a G-space with K a metrizable compact convex subset of a locally convex topological vector space such that each α^t: $K \to K$ is an affine mapping. Let $\langle X, \pi\rangle$ be a G-space and A a closed invariant subset such that every continuous function from A into $[0;1]$ has a continuous extension over all of X. Then every morphism of G-spaces*

$\underline{f}$: $\langle A, \pi\rangle \to \langle K, \alpha\rangle$ *has an extension to a morphism of G-spaces* $\bar{f}$: $\langle X, \pi\rangle \to \langle K, \alpha\rangle$.

REMARK. If X is metrizable, then the result also holds if K is not metrizable. In this case K need not even be compact: completeness is sufficient as was observed in ANTONYAN [4]. Variations of this result are surveyed in Section 2 of ANTONYAN [7]; also references to ANTONYAN [8], MADIRIMOV [15] and MURAYAMA [23] could be added. The results mentioned there are all for the case of a compact, or even finite, group G. For non-compact groups results are much more difficult to obtain. Let me mention the following (see DE VRIES [41]):

THEOREM. *Let* $\langle X, \pi\rangle$ *be a G-space with X a compact Hausdorff space, and let A be a closed invariant subset of X. The following conditions are equivalent:*
(i) *For every equicontinuous G-space* $\langle K, \alpha\rangle$ *with K and* α *as in the preceding theorem and for every morphism of G-spaces* $\underline{f}$: $\langle A, \pi\rangle \to \langle K, \alpha\rangle$ *there is an extension to a morphism of G-spaces* $\bar{f}$: $\langle X, \pi\rangle \to \langle K, \alpha\rangle$.
(ii) *Every almost periodic function* $\underline{h}$: $A \to \mathbb{R}$, *can be extended to an almost periodic function* $\bar{h}$: $X \to \mathbb{R}$.

(N.B. A G-space $\langle K, \alpha\rangle$ is said to be *equicontinuous* whenever $\{\alpha^t : t \in G\}$ is an equicontinuous family of self-maps of K with respect to the unique uniformity of K. If $\langle Y, \sigma\rangle$ is a G-space then a continuous function g: $Y \to \mathbb{R}$ is called *almost periodic* whenever g is bounded and the set $\{g \circ \sigma^t : t \in G\}$ is totally bounded with respect to the supremum norm.) The disadvantage of this result is that it characterizes pairs $(A, \langle X, \pi\rangle)$ (with A a closed invariant subset of the G-space $\langle X, \pi\rangle$) for which any G-spaces $\langle K, \alpha\rangle$ as indicated in part (i) of the theorem is a G-extensor in terms of an extension property of certain (real-valued) functions. There is yet another characterization, not using extension of functions; it uses terminology from topological dynamics, too complicated to explain here.

An amusing consequence of this theorem is the following proof that $\langle K, \alpha\rangle$ as above has an invariant point: embed $\langle K, \alpha\rangle$ equivariantly in $\langle K^*, \alpha^*\rangle$, where K^* is K to which one isolated invariant point is added; condition (ii) is trivially fulfilled, hence (i) implies that $\langle K, \alpha\rangle$ is an equivariant retract of $\langle K^*, \alpha^*\rangle$. The image in K of the invariant point of K^* is invariant in K. (It is well-known that $\langle K, \alpha\rangle$ as above has an invariant point even if K is not metrizable.)

Let me close this section by the observation that knowledge of G-extensors (which are absolute G-retracts!) is useful for the development of equivariant shape theory; cf. SMIRNOV [27,28] and a forthcoming paper by Antonyan and Mardesič. Up to now, the only satisfactory results in this area are for the case of a compact group.

3. LINEARIZATION

It is easy to show that if G is locally compact, then every Tychonov G-space $\langle X, \pi \rangle$ can be equivariantly embedded in a *linear G-space*, i.e. in a G-space $\langle V, \sigma \rangle$ with V a locally convex topological vector space such that each $\sigma^t: V \to V$ is linear. See e.g. DE VRIES [31] or SMIRNOV [25] (but at that moment this fact was already folklore). Problem: if X belongs to a certain special class of spaces, can V also be chosen from some related special class? For example, if X is metrizable, can V be chosen to be a Hilbert space? (Answer: yes, provided G is also σ-compact; cf. DE VRIES [31], which generalizes earlier work of Baayen and de Groot). Or: if X is a finite dimensional separable metric space, can V be chosen to be a Euclidean space? (Answer: if G is a compact Lie group and if $\langle X, \pi \rangle$ has only finitely many orbit types, then the answer is yes; cf. MOSTOV [22], and for later refinements, ALLAN [2] and the references given there.) For a survey of many linearization results, cf. DE VRIES [35], to which might be added JAWOROWSKI [13], MCCANN [16] (and for semiflows: MCCANN [17]), M.G. MEGRELISHVILI [21] and the references given there.

The following result generalizes (slightly) the main theorem of ANTONYAN [5] and results from DE VRIES [35]. It is based on the compactification theorem from Section 4 below and on some basic results from infinite dimensional topology:

THEOREM (ANTONYAN & DE VRIES [10]). *Let G be locally compact and σ-compact. Then for every cardinal number $\tau \geq w(G)$ there exists an action $\tilde{\pi}$ of G on $\mathbb{R}^\tau$ such that:*

(i) *the cube $[0;1]^\tau$ is an invariant subset of $\mathbb{R}^\tau$;*

(ii) *every G-space $\langle X, \pi \rangle$ with X a Tychonov space of weight $w(X) \leq \tau$ can be equivariantly embedded in $[0;1]^\tau$.*

Moreover, there exists a linear structure on $\mathbb{R}^\tau$ such that $\mathbb{R}^\tau$ is a locally convex topological vector space and

(iii) *the action $\tilde{\pi}$ is linear (i.e. each $\tilde{\pi}^t$ is linear);*

(iv) *$[0;1]^\tau$ is a convex subset of $\mathbb{R}^\tau$.*

4. COMPACTIFICATION

From general categorical considerations it follows easily that $\mathbf{Comp}^G$ is a reflective subcategory of $\mathbf{Top}^G$. For a G-space $\langle X, \pi \rangle$ let its reflection in $\mathbf{Comp}^G$ be denoted by

$$\phi_G: \langle X, \pi \rangle \to \langle \beta_G X, \pi \rangle.$$

Question: if X is a Tychonov space, is ϕ_G an equivariant dense *embedding*? Answer: yes, provided G is locally compact. This is an immediate consequence of the following

THEOREM. *Let G be locally compact and let $\langle X, \pi \rangle$ be a G-space with X a Tychonov space. Then $\langle X, \pi \rangle$ can be equivariantly embedded in a G-space $\langle Y, \sigma \rangle$*

with Y a compact Hausdoff space of weight $w(Y) \leq \max\{\mathfrak{L}(G/G_0), w(X)\}$. *(Here* $G_0 := \{t \in G: \pi^t = id_X\}$ *and* $\mathfrak{L}(G/G_0)$ *is the Lindelöf degree of* G/G_0.*)*

REMARKS. For two different proofs, cf. DE VRIES [33,34]; a third proof for the case that G is compact was obtained independently in ANTONYAN [6]; see also ANTONYAN [3]. Yet another proof was obtained by M.G. Megrelishvili: every G-space admitting a G-linearization has a G-compactification; see M.G. MEGRELISHVILI [20,21]. For the inequality concerning the weight of a possible equivariant compactification, cf. DE VRIES [36]. Observe that for the existence of an equivariant embedding in a compact G-space, local compactness of G is not strictly needed: it is sufficient that there exists a uniformity in X with respect to which some neighbourhood of e in G acts equicontinuously; cf. DE VRIES [37] and also MEGRELISHVILI [19]. An example in ANTONYAN & SMIRNOV [9] shows, however, that some additional condition is needed; see also M.G. MEGRELISHVILI [21].

The equivariant embedding ϕ_G: $\langle X, \pi\rangle \rightarrow \langle \beta_G X, \pi\rangle$ (it will be assumed that G is locally compact and all G-spaces are Tychonov) plays the role of the Stone-Čeck compactification from **Top**. Explicit examples are contained in SMIRNOV & STOYANOV [29] and STOYANOV [30]. In DE VRIES [38], an equivariant version of Glicksberg's theorem was obtained. Using results from DE VRIES [39], this theorem can now be formulated as follows: provided a certain 'non-triviality' condition is fulfilled one has $\beta_G(\Pi_\alpha X_\alpha) = \Pi_\alpha \beta_G X_\alpha$ if and only if $\Pi_\alpha X_\alpha$ is pseudocompact (for the 'only if' part, G must also be assumed to be locally connected). See DE VRIES [40].

5. OTHER RESULTS

Interesting results concerning among other things dimension theory have been obtained by Smirnov and his co-workers. See SMIRNOV [26], AGEEV [1], BALADZE [11] and MEGRELISHVILI [18]. For equivariant completions, see MEGRELISHVILI [19, 20].

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